

# A NOTE ON A CERTAIN BAUM–CONNES MAP FOR INVERSE SEMIGROUPS

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ABSTRACT. Let  $G$  denote a countable inverse semigroup. We construct a kind of a Baum–Connes map  $K(\tilde{A} \rtimes G) \rightarrow K(A \rtimes G)$  by a categorical approach via localization of triangulated categories, developed by R. Meyer and R. Nest for groups  $G$ . We allow the coefficient algebras  $A$  to be in a special class of algebras called fibered  $G$ -algebras. This note continues and fixes our preprint “Attempts to define a Baum–Connes map via localization of categories for inverse semigroups”.

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## 1. INTRODUCTION

In [14], R. Meyer and R. Nest found an equivalent definition of the Baum–Connes map [1]. The new definition defines the Baum–Connes map for a coefficient  $G$ -algebra  $A$  as a homomorphism

$$K(\tilde{A} \rtimes G) \rightarrow K(A \rtimes G)$$

of  $K$ -theory groups, where  $\tilde{A}$  is a certain approximation for  $A$ . To be precise,  $\tilde{A}$  sits in the triangulated subcategory of  $KK^G$  which is generated by induced algebras of the form  $\text{Ind}_H^G(B)$  for some compact subgroups  $H$  of  $G$ . This gives a potentially

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possible way to compute the left hand side  $K(\tilde{A} \rtimes G)$  of the new Baum–Connes map by homological means. If  $\tilde{A} = \text{Ind}_H^G(B)$  then one may directly use

$$(1) \quad K(\tilde{A} \rtimes G) = K(\text{Ind}_H^G(B) \rtimes G) \cong K(B \rtimes H) \cong K^H(\mathbb{C}, B)$$

by Green’s imprimitivity theorem [6] and the Green–Julg isomorphism.

In [3] we tried to define a Baum–Connes map in an analogous way for inverse semigroups  $G$ , but ran into some problems which at the end turned out to rely on the fact that we assumed a wrong right adjoint functor to the induction functor.

In this note we shall define a correct right adjoint functor, called the fibered restriction functor, and is given by  $R_G^H(A) = \bigoplus_{e \in E_H} A_{\varepsilon_e}$ . It turns out, however, that it will only work on the subcategory of  $KK^G$  which is generated by fibered  $G$ -algebras. These are  $G$ -algebras of the form  $\bigoplus_{e \in E} A_{\varepsilon_e}$ . ( $C_0(X)$  is not fibered.) Hence we can build a Baum–Connes map only for fibered coefficient algebras.

The idea is as follows. We analyzed the  $G$ -action on  $\text{Ind}_H^G(A)$  and interpreted it as a groupoid action. Then we made the simple observation (encoded in Lemma 3.1) that a simple characteristic function  $1_{g(1-e_1)\dots(1-e_n)} \in \text{Ind}_H^G(A)$  for  $g \in G, e_i \in E$  has carrier the single point  $\varepsilon_{gg^*}$  in the base space of the groupoid. (The character is defined by  $\varepsilon_e(f) = 1_{\{f \geq e\}}$ .) The  $G$ -action on  $\text{Ind}_H^G(A)$  then just shifts  $1_g$  as usual, that is,  $h(1_g)$  has carrier the single point  $\varepsilon_{hgg^*h^*}$  for  $h \in G$ . In other words,  $\text{Ind}_H^G(A)$  is a fibered  $G$ -algebra. We shall not lay out all the heuristical idea, but it is encoded in this paper and see Remark 5.2 and Example 7.3.

The main work of this note is the definition of the fibered restriction functor in 6.1 and the proof that it is right adjoint to the induction functor in 7.2. We shall also introduce a new  $\ell^2(G)$ -space as a technical tool in 5.5. We can use blueprints from [3] to obtain the fact that  $KK^G$  is triangulated for fibered  $G$ -algebras. (This is of course analogous to the proof by [12] and [14].) Fortunately, Ralf Meyer pointed out to us his work [13] soon after publishing [3], which then immediately yields the approximation  $\tilde{A}$  mentioned above and thus the Baum–Connes map, and we need not go through the technicalities to define such an approximation as in [3].

The resulting Baum–Connes map given in Definition 10.14 is justified in so far as computation (1) works also for inverse semigroups and Sieben’s crossed product [18] by 4.5. It has, however, usually less potential for computing the full crossed product  $\mathbb{C} \rtimes G$ , see Remark 10.15.

Sections 3–7 essentially occupy the definition of the fibered restriction functor and the verification that it is right adjoint to the induction functor. The last Sections 8–10 cover slight adaption and collection of known results.

## 2. SOME NOTATION

In this note,  $G$  denotes a countable discrete inverse semigroup. We define  $G$ -actions on  $C^*$ -algebras and  $G$ -equivariant  $KK$ -theory  $KK^G$  as in [5]. In case of an inverse semigroup  $G$  the formal definitions simplify slightly, for which we refer to [3]. The letter  $C_G^*$  will denote the category of  $G$ -algebras with their  $G$ -equivariant  $*$ -homomorphisms.  $KK^G$  will stand for the Kasparov category consisting of  $G$ -algebras as object class and Kasparov groups as morphism sets. For convenience of the reader we recall the notion of a  $G$ -action on a  $C^*$ -algebra.

**Definition 2.1.** A  $G$ -algebra is a  $C^*$ -algebra  $A$  equipped with a semigroup homomorphism  $\alpha : G \rightarrow \text{End}(A)$  such that  $\alpha_e(a)b = a\alpha_e(b)$  for all  $a, b \in A$  and  $e \in E$ .

Throughout we write  $g(a) := \alpha_g(a)$ . The letter  $E$  stands for the set of idempotent elements of  $G$ , and  $C^*(E)$  for the abelian  $C^*$ -algebra it generates. We write  $C^*(E) \cong C_0(X)$  by Gelfand's theorem, where  $X$  denotes the character space of  $C^*(E)$ . Every  $e \in E$  defines a character  $\varepsilon_e \in X$  by  $\varepsilon_e(f) = 1_{\{f \geq e\}}$  for all  $f \in E$ . The character set  $\{\varepsilon_e\}_{e \in E}$  is dense in  $X$ . See [17] or [9, 2.6] for more details.

The algebra  $C^*(E)$  is a  $G$ -algebra under the  $G$ -action  $g(e) = geg^*$ . For heuristical comments we shall consider a bigger function space than  $C_0(X)$ . Let  $\mathbb{C}^X$  be the set of functions from  $X$  to  $\mathbb{C}$ . It is endowed with the  $G$ -action induced by the maps  $g : X \rightarrow X$  given by  $(g(x))(e) = x(g^*eg)$  for all  $x \in X, g \in G, e \in E$ . This is consistent with the  $G$ -action on  $C_0(X)$  defined before, that is,  $C_0(X) \subseteq \mathbb{C}^X$   $G$ -equivariantly. We shall write  $1_x$  for the characteristic function  $1_{\{x\}}$  of a single point  $x$ .

Every  $G$ -algebra  $A$  is equipped with a  $G$ -equivariant  $*$ -homomorphism  $C_0(X) \rightarrow \mathcal{ZM}(A)$  (center of the multiplier algebra of  $A$ ). We denote the  $C_0(X)$ -balanced tensor product of  $G$ -algebras or  $C_0(X)$ -algebras  $A, B$  by  $A \otimes^{C_0(X)} B := (A \otimes B)/I$ , where  $I$  is the ideal generated by  $e(a) \otimes b - a \otimes e(b)$  for all  $a, b \in A, e \in E$ . We write  $A_x = \mathbb{C}\{1_x\} \otimes^{C_0(X)} A$  for the fiber of  $A$  in  $x \in X$ .

The universal (or occasionally an unspecified) crossed product [9] is denoted by  $A \rtimes G$ , and Sieben's ("compatible" universal) crossed product [18] by  $A \widehat{\rtimes} G$ . We sometimes consider another inverse semigroups

$$\begin{aligned} \tilde{E} &= \{e_0(1 - e_1) \cdots (1 - e_n) \in \mathbb{C} \rtimes G \mid e_i \in E, n \geq 1\}, \\ \tilde{G} &= \{gp \in \mathbb{C} \rtimes G \mid g \in G, p \in \tilde{E}\} \end{aligned}$$

as subinverse semigroups of  $\mathbb{C} \rtimes G$  under multiplication and involution.

For an assertion  $\mathcal{A}$  we write  $[\mathcal{A}]$  for the real number which is 1 if  $\mathcal{A}$  is true, and 0 otherwise. In case that we have given another inverse semigroup  $H$  we shall specify the associated sets  $E$  and  $X$  by writing  $E_H$  and  $X_H$ . Usually  $H$  denotes a finite subinverse semigroup of  $G$  and  $\text{Res}_G^H$  the usual restriction functor. For possibly further needed details we refer to [3].

### 3. SOME LEMMAS

In this section we observe some central lemmas.

**Lemma 3.1.** *Let  $f \in E$  and  $p := e_0(1 - e_1) \cdots (1 - e_n)$  a nonzero element in  $\tilde{E}$  ( $e_i \in E$ ). Then*

$$f \geq p \iff f \geq e_0.$$

*Proof.*  $\Leftarrow$  is clear.  $\Rightarrow$ : By expanding the brackets in  $fp = p$  we get

$$(2) \quad fe_0 - fe_0e_1 + fe_0e_1e_2 \pm \dots = e_0 - e_0e_1 + e_0e_1e_2 \pm \dots$$

Since  $e_0(1 - e_1) \cdots (1 - e_n) \neq 0$ ,  $e_0e_i \neq e_0$  for all  $i \geq 1$ . Hence

$$e_0e_1, e_0e_1e_2, e_0e_2, \dots < e_0.$$

Since the projections  $E$  are linearly independent in  $C^*(E)$ , the only possibility that (2) is true is that  $fe_0 = e_0$ . That is,  $f \geq e_0$ .  $\square$

**Definition 3.2.** For a nonzero  $p = e_0(1 - e_1) \cdots (1 - e_n) \in \tilde{E}$  ( $e_i \in E$ ) set

$$\sigma(p) := e_0,$$

the *leading coefficient* of  $p$ . This yields a map  $\sigma : \tilde{E} \setminus \{0\} \rightarrow E$ .

Note that the last definition is well-defined since  $e_0$  is the unique minimal projection in  $E$  such that  $e_0 \geq p$  by Lemma 3.1. The next lemma shows how the leading coefficient already uniquely determines certain sets of projections in  $\tilde{E}$ .

**Lemma 3.3.** *Let  $H \subseteq G$  be a finite subinverse semigroup. View  $X_H \subseteq \tilde{E}_H$  as the set of all minimal nonzero projections of  $\tilde{E}_H$ . Then the map  $\sigma|_{X_H} : X_H \rightarrow E_H$  is a bijection.*

*Proof.* (a) Let  $p = e_0(1 - e_1) \dots (1 - e_n)$  and  $q = f_0(1 - f_1) \dots (1 - f_m)$  be nonzero elements in  $X_H$  for  $e_i, f_j \in E_H$ . Assume that  $\sigma(p) = e_0 = f_0 = \sigma(q)$ . Let  $i \geq 1$ . Note that  $e_i \not\geq e_0 = f_0$ . Thus  $e_i f_0 < f_0$ . Hence, necessarily  $(1 - e_i)q = q$ . Consequently  $p \geq q$ . Similarly  $p \leq q$ . This shows injectivity of  $\sigma$ . To prove surjectivity, consider  $e \in E_H$ . Set  $p = e \prod_{f \in E_H, f \not\geq e} (1 - f)$ . Then  $p \in X_H$  and  $\sigma(p) = e$ .  $\square$

The restriction  $\sigma|_{X_H}$  will also be denoted by  $\sigma_H$ . Since  $H_H = H$  as sets (see Definition 4.1 below), we shall also write  $H/H$  for  $H_H/H$ .

**Lemma 3.4.** (a) *One has  $\sigma(p) \geq p$  for all  $p \in \tilde{E}$ .*  
 (b) *We have  $\sigma(gpg^*) = g\sigma(p)g^*$  for all  $g \in G$  and  $p \in \tilde{E}$ .*  
 (c) *Furthermore,  $\sigma_H^{-1}(e) \leq e$  and  $\sigma_H^{-1}(heh^*) = h\sigma_H^{-1}(e)h^*$  for all  $h \in H$  and  $e \in E_H$ .*  
 (d) *Moreover,  $\sum_{h \in H/H} \sigma_H^{-1}(hh^*) = 1_H$ .*  
 (e) *The map  $\sigma$  is multiplicative.*

*Proof.* (a) and (e) are clear. (b) Since  $gpg^* = ge_0g^*(1 - ge_1g^*) \dots (1 - ge_ng^*)$ . (c) follow from (b). (d) Note that every element of  $E_H$  is of the form  $hh^*$  for some  $h \in H$ , and  $h_1 \equiv h_2$  for  $h_i \in H$  if and only if  $h_1h_1^* = h_2h_2^*$  (if and only if  $h_1 = h_2h_2^*h_1$ ). Hence  $\sum_{h \in H/H} \sigma_H^{-1}(hh^*) = \sum_{p \in X_H} p = 1_H$  by Lemma 3.3.  $\square$

#### 4. THE INDUCTION FUNCTOR

In this section we recall the definition of a  $G$ -algebra which is induced by an  $H$ -algebra for a finite subinverse semigroup  $H$  of  $G$ . We shall use a formally slightly modified but equivalent definition than in [2], see Corollary 4.7. The reason is the observation made in Lemma 4.6 that we may change slightly a set called  $G_H$ .

**Definition 4.1.** Let  $H \subseteq G$  be a finite subinverse semigroup. We define  $G_H \subseteq G$  as

$$G_H := \{g \in G \mid g^*g \in H\}.$$

We endow  $G_H$  with an equivalence relation:  $g \equiv l$  ( $g, l \in G_H$ ) if and only if  $gh = l$  for some  $h \in H$  with  $g^*g = hh^*$ . The set-theoretical quotient  $G_H / \equiv$  is denoted by  $G_H/H$ .

**Definition 4.2.** Let  $c_0(G)$  be the usual commutative  $C^*$ -algebra of complex-valued functions on  $G$  vanishing at infinity ( $G$  being discrete), endowed with the  $G$ -action  $(k(f))(g) = f(k^*g) [kk^* \geq gg^*]$  for all  $f \in c_0(G)$  and  $k, g \in G$ . This turns  $c_0(G)$  to a  $G$ -algebra.

We denote by  $c_0(G_H)$  the  $G$ -subalgebra of  $c_0(G)$  consisting of all functions vanishing outside of  $G_H$ . We similarly define  $c_0(G_H/H)$  as the usual commutative  $C^*$ -algebra, endowed with the  $G$ -action  $(k(f))(gH) = f(k^*gH) [kk^* \geq gg^*]$  for all  $f \in c_0(G_H/H)$  and  $k, g \in G$ , which turns it to a  $G$ -algebra.

**Definition 4.3.** Let  $H \subseteq G$  be a finite subinversive semigroup and  $D$  a  $H$ -algebra. Define, similar as in [8, §5 Def. 2],

$$\begin{aligned} \text{Ind}_H^G(D) &:= \{f : G_H \rightarrow D \mid \forall g \in G_H : \forall h \in H \text{ with } g^*g = hh^* : \\ &\quad f(gh) = \sigma_H^{-1}(h^*h) h^*(f(g)), \quad \|f(g)\| \rightarrow 0 \text{ for } gh \rightarrow \infty\}. \end{aligned}$$

It is a  $C^*$ -algebra under the pointwise operations and the supremum's norm  $\|f\| = \sup_{g \in G_H} \|f(g)\|$ , and becomes a  $G$ -algebra under the  $G$ -action  $(k(f))(h) := f(k^*g) [kk^* \geq gg^*]$  for all  $k \in G, g \in G_H$  and  $f \in \text{Ind}_H^G(D)$ .

**Definition 4.4.** Let  $H \subseteq G$  be a finite subinversive semigroup. By the universal property of  $KK^H$ -theory [4], there exists an *induction functor*  $\text{Ind}_H^G : KK^H \rightarrow KK^G$  induced by the functor  $F : C_H^* \rightarrow C_G^*$  given by  $F(A) = \text{Ind}_H^G(A)$  for  $H$ -algebras  $A$ . For more details see [3].

A key motivation for the definition of an induction algebra is the following variant of Green's imprimitivity theorem [6].

**Theorem 4.5** ([2]). *Let  $H \subseteq G$  be a finite subinversive semigroup and  $A$  a  $H$ -algebra. Then the  $C^*$ -algebras  $\text{Ind}_H^G(A) \widehat{\rtimes} G$  and  $A \widehat{\rtimes} H$  are Morita equivalent.*

In [2] we defined  $G_H$  and  $\text{Ind}_H^G(D)$  slightly differently. But as already mentioned, both definitions are equivalent. To explain this in detail, re-denote  $G_H$  of [2] as  $G'_H$ . Define  $\mathcal{G}$  as the finite groupoid associated to  $H$ . That is, one sets  $\mathcal{G}^{(0)} := X_H \subseteq \tilde{E}_H \subseteq \tilde{G}$  and  $\mathcal{G} = \{hp \in \tilde{G} \mid h \in H, p \in \mathcal{G}^{(0)}\} \setminus \{0\} \subseteq \tilde{G}$ .

One then sets  $G'_H = \{t \in \tilde{G} \mid t^*t \in \mathcal{G}^{(0)}\}$ . An equivalence relation on  $G'_H$  is given by  $t \equiv s$  ( $t, s \in G'_H$ ) if and only if there exists an  $h \in \mathcal{G}$  (equivalently:  $h \in H$ ) such that  $th = s$ .

**Lemma 4.6.** *The map  $\delta : G'_H \rightarrow G_H$  given by  $\delta(gp) = g\sigma(p)$  ( $g \in G, p \in \mathcal{G}^{(0)}$ ) is a bijection which respects the equivalence relations in both directions.*

*The inverse map is given by  $\delta^{-1}(g) = g\sigma_H^{-1}(g^*g)$  for all  $g \in G_H$ .*

*Proof.* Given  $gp \in G'_H$  we observe that

$$(3) \quad g^*g \geq \sigma(p) \geq p.$$

Indeed, notice that  $g^*g \geq p$  because by definition  $(gp)^*(gp)$  is in  $X_H$  and so can only be  $p$ . Thus by Lemma 3.1,  $g^*g \geq \sigma(p)$ . It is then straightforward to check with Lemma 3.1 that  $\delta$  and  $\delta^{-1}$  are inverses to each other.

We are going to discuss the equivalence relations. Suppose that  $gp \equiv g'p'$  in  $G'_H$ . Then there exists a  $h \in H$  such that  $gph = g'p'$ , or  $gh(h^*ph) = g'p'$ , and  $hh^* \geq p$  (recalling (3)). Applying  $\delta$ , we get  $gh\sigma(h^*ph) = g\sigma(p)h = g'\sigma(p)$  with Lemma 3.4. In other words,  $\delta(gp)h' = \delta(g'p')$  for  $h' := \sigma(p)h \in H$ . By Lemma 3.1,  $hh^* \geq \sigma(p)$ , and thus  $h'h'^* = \delta(gp)\delta(gp)^*$ . Hence  $\delta(gp) \equiv \delta(g'p')$  in  $G_H$ .

If  $g \equiv g'$  in  $G_H$  then there exists a  $h \in H$  with  $gh = g'$  and  $g^*g = hh^*$ . Recall that  $\delta^{-1}(g) = gp$  with  $\sigma(p) = g^*g$ . Hence  $\delta^{-1}(gh) = \delta^{-1}(g') = ghp = g'p'$  with  $h^*h = \sigma(p)$  and  $g'^*g' = \sigma(p')$ . Notice that  $g^*g = hh^* = \sigma(hph^*)$  with Lemma 3.4. Consequently  $\delta^{-1}(g)h = ghph^*h = \delta^{-1}(g')$ . Hence  $\delta^{-1}(g) \equiv \delta^{-1}(g')$  in  $G'_H$ .  $\square$

Let us re-denote the induction algebra  $\text{Ind}_H^G(D)$  of [2] as  $\text{Ind}_H^G(D)'$ .

**Corollary 4.7.** *There is a  $G$ -equivariant isomorphism  $\text{Ind}_H^G(A) \rightarrow \text{Ind}_H^G(A)'$ .*

*Proof.* The isomorphism  $\varphi$  is given by  $\varphi(f)(t) = f(\delta(t))$  for all  $f \in \text{Ind}_H^G(A)$  and  $t \in G'_H$ . To see that it is well-defined, consider  $t = gp$  and  $h = kp \in \mathcal{G}$  for  $g \in G, p \in X_H, k \in H$ . Note that  $g^*g, kk^* \geq p$  and thus  $g^*g, kk^* \geq \sigma(p)$  by Lemma 3.1. Then  $(\varphi(f))(th) = f(\delta(th)) = f(\delta(gkk^*pk)) = f(gk\sigma(k^*pk)) = f(g\sigma(p)k) = \sigma_H^{-1}(k^*\sigma(p)k)k^*\sigma(p)f(g\sigma(p)) = k^*pf(\delta(t)) = h^*(\varphi(f)(t))$  by Lemma 3.4.

Surjectivity may be observed by setting  $h = p$  and  $k = 1$  in the last computation. For the  $G$ -invariance just notice that an  $f \in E$  satisfies  $f \geq gpg^*$  if and only if  $f \geq \sigma(gpg^*) = g\sigma(p)g^*$  by Lemma 3.1.  $\square$

## 5. THE $\ell^2(G)$ -SPACE

In this section we will shall define fibered  $G$ -algebras and an  $\ell^2(G)$ -space as a tool for working with such algebras.

**Definition 5.1.** Let  $\varepsilon(E)$  denote the commutative  $C^*$ -algebra  $c_0(E)$  ( $E$  being discrete). We turn  $\varepsilon(E)$  to a  $G$ -algebra by setting  $(g(f))(e) = f(g^*eg) [gg^* \geq e]$  for all  $g \in G, f \in \varepsilon(E)$  and  $e \in E$ .

Equivalently we may define the  $G$ -action by  $g(1_e) = 1_{geg^*} [gg^* \geq e]$  for all  $e \in E$  ( $1_e := 1_{\{e\}} \in \varepsilon(E)$ ). The algebra  $\varepsilon(E)$  will be used as a replacement for  $\mathbb{C}$  as utilized in group equivariant  $C^*$ -theory, see Lemma 5.7.

**Remark 5.2.** Heuristically and even exactly if we like, we view the characteristic function  $1_e \in \varepsilon(E)$  as the characteristic function  $1_{\varepsilon_e} \in \mathbb{C}^X$  of the point  $\varepsilon_e \in X$ . In other words,  $\varepsilon(E)$  is  $G$ -equivariantly  $*$ -isomorphic to the  $G$ -invariant  $G$ -subalgebra of  $\mathbb{C}^X$  generated by the simple functions  $1_{\varepsilon_e}$  via the map  $1_e \mapsto 1_{\varepsilon_e}$ .

Fibers of a  $C_0(X)$ -algebra  $A$  may be computed by

$$A_{\varepsilon_e} = \mathbb{C}\{1_e\} \otimes^{C_0(X)} A.$$

Consequently

$$(4) \quad \bigoplus_{e \in E} A_{\varepsilon_e} = \varepsilon(E) \otimes^{C_0(X)} A.$$

**Lemma 5.3.**  $\varepsilon(E)$  is a  $G$ -algebra.

*Proof.* We have  $(gh(f))(e) = f(h^*g^*egh) [ghh^*g^* \geq e]$  and  $(g(h(f)))(e) = f(h^*g^*egh) [gg^* \geq e] [hh^* \geq g^*eg]$  for  $g, h \in G, e \in E$  and  $f \in \varepsilon(E)$ . Now  $ghh^*g^* \geq e$  implies  $hh^* \geq g^*ghh^*g^*g \geq g^*eg$  and  $gg^* \geq ghh^*g^* \geq e$  and similarly reversely. Hence  $(gh)(f) = g(h(f))$ . Further,  $(e_1(f_1)f_2)(e_2) = f_1(e_1e_2)f_2(e_2) [e_1 \geq e_2] = (f_1 \cdot e_1(f_2))(e_2)$  for  $e_i \in E$ .  $\square$

**Lemma 5.4.** Let  $h \in G$ . There are bijections

$$\begin{aligned} \{g \in G \mid gg^* = h^*h\} &\rightarrow \{g \in G \mid gg^* = hh^*\} : g \mapsto hg, \\ \{g \in G \mid gg^* \leq h^*h\} &\rightarrow \{g \in G \mid gg^* \leq hh^*\} : g \mapsto hg. \end{aligned}$$

Similar things can be said for the right multiplication  $g \mapsto gh$ .

**Definition 5.5.** Let  $c_c(G)$  denote the linear space consisting of functions  $G \rightarrow \mathbb{C}$  with finite support. We turn  $c_c(G)$  to a right  $\varepsilon(E)$ -module by setting

$$(\xi f)(g) = \xi(g)f(gg^*)$$

for all  $\xi \in c_c(G)$ ,  $f \in c_0(E)$  and  $g \in G$ . This module is endowed with an  $\varepsilon(E)$ -valued inner product given by

$$\langle \xi, \eta \rangle(e) = \sum_{g \in G, gg^* = e} \overline{\xi(g)} \eta(g).$$

The space  $c_c(G)$  will be equipped with the  $G$ -action

$$(h\xi)(g) = \xi(h^*g) [hh^* \geq gg^*]$$

for all  $\xi \in c_c(G)$  and  $h, g \in G$ .

The closure of  $c_c(G)$  under the norm induced by the inner product is a  $G$ -Hilbert  $\varepsilon(E)$ -module denoted by  $\ell^2(G)$ .

**Lemma 5.6.** *The space  $\ell^2(G)$  is a  $G$ -Hilbert  $\varepsilon(E)$ -module.*

*Proof.* It is obvious that the inner product is positive definite. The module structure is straightforward to check, and for the admissibility of the  $G$ -action confer the proof of Lemma 5.3. It is well-known that  $\ell^2(G)$  is consequently a Hilbert  $\varepsilon(E)$ -module. Also the  $G$ -action extends to  $\ell^2(G)$  by continuity of linear operators:

$$\|h(\xi)\|_{c_c(G)}^2 = \sup_{e \in E} \left\| \sum_{g \in G, gg^* = e} \overline{\xi(h^*g)} \eta(h^*g) [hh^* \geq e] \right\| \leq \|\xi\|_{c_c(G)}^2$$

by Lemma 5.4. □

Similarly to  $\ell^2(G)$  we may define a  $G$ -Hilbert  $\varepsilon(E)$ -module  $\ell^2(G_H/H)$ . (It may be regarded as the submodule of  $\ell^2(G)$  consisting of all functions vanishing outside of  $G_H$  and being constant on equivalence classes.)

**Lemma 5.7.** *There are  $G$ -equivariant  $*$ -isomorphisms*

$$A \otimes^{C_0(X)} \varepsilon(E) \cong A$$

for the  $G$ -algebras  $A = \varepsilon(E), c_0(G), c_0(G_H), c_0(G_H/H), \text{Ind}_H^G(D)$ .

*Proof.* One checks that  $a \otimes b \mapsto ab$  realizes these isomorphisms, where  $ab$  is the module multiplication used in Definition 5.5. □

**Definition 5.8.** A *fibred  $G$ -algebra* is a  $G$ -algebra of the form  $\varepsilon(E) \otimes^{C_0(X)} A$  up to isomorphism for some  $G$ -algebra  $A$ .

If  $G$  is finite then every  $G$ -algebra is fibred. Indeed, (4) is  $A$  by the fact that  $\varepsilon_E = X$ .

## 6. THE FIBERED RESTRICTION FUNCTOR

Let  $H$  be a finite subinverse semigroup of  $G$ . Regard  $\varepsilon(E_H)$  as a  $G$ -subalgebra of  $\varepsilon(E)$  in a canonical way. We shall denote by  $H \cdot E \subseteq G$  the subinverse semigroup of  $G$  generated by  $H$  and  $E$ . Note that  $\varepsilon(E_H)$  is a  $H \cdot E$ -invariant subalgebra of  $\varepsilon(E)$ .

We define the fibred restriction as  $R_G^H(A) = \oplus_{e \in E_H} A_{\varepsilon_e}$  for  $G$ -algebras  $A$ . In another way me say:

**Definition 6.1.** Let  $H \subseteq G$  be a finite subinverse semigroup. Define the *fibred restriction functor*  $R_G^H : KK^G \rightarrow KK^H$  by

$$(5) \quad R_G^H(A) = \text{Res}_G^H(\varepsilon(E_H) \otimes^{C_0(X)} A)$$

$$(6) \quad := \text{Res}_{H \cdot E}^H(\varepsilon(E_H) \otimes^{C_0(X)} \text{Res}_G^{H \cdot E}(A))$$

for an object  $A$  in  $KK^G$ . The meaning of (5) is more precisely repeated in (6). As usual, for a morphism  $[(\pi, \mathcal{E}, T)] \in KK^G(A, B)$  one sets

$$R_G^H[(\pi, \mathcal{E}, T)] = [(1 \otimes \pi, \text{Res}_G^H(\varepsilon(E_H) \otimes^{C_0(X)} \mathcal{E}), 1 \otimes T)] \in KK^H(R_G^H(A), R_G^H(B)).$$

**Lemma 6.2.** *Let  $H \subseteq G$  be a finite subinverse semigroup and  $B$  a  $G$ -algebra. There is a  $G$ -equivariant isomorphism*

$$\mu : \text{Ind}_H^G R_G^H(B) \rightarrow c_0(G_H/H) \otimes^{C_0(X)} B$$

given by

$$\mu(f) = \sum_{g \in G_H/H} 1_{gH} \otimes g(f(g)) \in c_0(G_H/H) \otimes^{C_0(X)} \varepsilon(E) \otimes^{C_0(X)} B,$$

where  $f \in \text{Ind}_H^G R_G^H(B)$  is interpreted as a function  $f : G_H \rightarrow \varepsilon(E) \otimes^{C_0(X)} B$  and the isomorphism of Lemma 5.7 is used.

*Proof.* The map is well-defined since if  $gH = g'H$  then  $gh = g'h$  for some  $h \in H$  with  $g^*g = hh^*$  and so  $g'(f(g')) = gh(f(gh)) = gh\sigma_H^{-1}(h^*h)h^*(f(g)) = g\sigma_H^{-1}(hh^*)hh^*(f(g)) = g(f(ghh^*)) = g(f(g))$  with Lemma 3.4. Injectivity is indicated as  $0 = g(f(g))$  implies  $0 = \sigma_H^{-1}(g^*g)g^*g(f(g)) = f(gg^*g)$  for all  $g \in G_H$ . Surjectivity is because of  $\mu(\sum_{k \in gH} 1_k \otimes 1_{k^*k} \otimes k^*(a)) = 1_{gH} \otimes gg^*(b)$  for all  $g \in G_H, b \in B$ . The  $G$ -equivariance is computed by

$$\begin{aligned} \mu(k(f)) &= \sum_{g \in G_H/H} 1_{gH} \otimes g(f(k^*g)) [kk^* \geq gg^*] \\ &= \sum_{g \in G_H/H} 1_{kgH} \otimes kg(f(g)) [k^*k \geq gg^*] = k(\mu(f)) \end{aligned}$$

for all  $k \in G$  by Lemma 5.4.  $\square$

**Lemma 6.3.** *One has an isomorphism*

$$R_G^H(A \otimes^{C_0(X)} B) \cong R_G^H(A) \otimes^{C_0(X_H)} R_G^H(B)$$

for all  $G$ -algebras  $A$  and  $B$ .

The following lemma is similar to Lemma 6.2 with a similar proof.

**Lemma 6.4.** *Let  $H \subseteq G$  be a finite subinverse semigroup,  $A$  a  $H$ -algebra and  $B$  a  $G$ -algebra. Then there is a  $G$ -equivariant isomorphism*

$$\tau : \text{Ind}_H^G(A \otimes^{C_0(X_H)} R_G^H(B)) \rightarrow \text{Ind}_H^G(A) \otimes^{C_0(X)} B$$

induced by

$$\tau(1_g \otimes a \otimes b) = 1_g \otimes a \otimes g(b) \in \text{Ind}_G^H(A) \otimes^{C_0(X)} \varepsilon(E) \otimes^{C_0(X)} B,$$

where  $g \in G_H, a \in A, b \in R_G^H(B) \subseteq \varepsilon(E) \otimes^{C_0(X)} B$  and the isomorphism of Lemma 5.7 is used.



## 7. THE ADJOINTNESS RELATION

We recall a known result, whose general proof holds also unmodified in the inverse semigroup equivariant setting.

**Lemma 7.1.** *Every full  $G$ -Hilbert  $B$ -module  $\mathcal{H}$  is an imprimitivity bimodule which establishes a  $G$ -equivariant Morita equivalence between  $\mathcal{K}(\mathcal{H})$  and  $B$ . Hence the element  $[(\mathcal{H}, 0)] \in KK^G(\mathcal{K}(\mathcal{H}), B)$  is invertible.*

**Proposition 7.2.** *If we allow only fibered  $C^*$ -algebras then the functor  $\text{Ind}_H^G$  is left adjoint to the functor  $R_G^H$ . In other words, one has an isomorphism*

$$KK^G(\text{Ind}_H^G(A), B) \cong KK^H(A, R_G^H(B))$$

which is natural in  $A$  and  $B$  for all  $H$ -algebra  $A$  and fibered  $G$ -algebras  $B$ .

*Proof.* In this proof we restrict  $C_G^*$  and the object class of  $KK^G$  to fibered  $G$ -algebras.

We shall consider two projections of adjunction. One is the transformation  $\iota$  of the functors  $\text{id}_{C_H^*}$  and  $R_G^H \text{Ind}_H^G$  given by the family of homomorphisms

$$\iota_A : A \rightarrow R_G^H \text{Ind}_H^G(A), \quad \iota_A(a) = 1_{\varepsilon(E_H)} \otimes (g \mapsto \sigma_H^{-1}(g^*g)g^*(a) [g \in H])$$

for  $A$  in  $C_H^*$  and  $a \in A, g \in G_H$ . In other words, we may say that

$$(7) \quad \iota_A(a) = \sum_{h \in H} 1_{hh^*} \otimes 1_h \otimes \sigma_H^{-1}(h^*h)h^*(a)$$

$$(8) \quad = \sum_{h \in H/H} 1_{hh^*} \otimes \sum_{k \in hH} 1_k \otimes \sigma_H^{-1}(k^*k)k^*(a).$$

Line (7) is the imprecise notation as the summands are actually not elements of  $R_G^H \text{Ind}_H^G(A)$ , and (8) is the correct meaning.

Two is the transformation  $\pi$  of the functors  $\text{Ind}_H^G R_G^H$  and  $\text{id}_{C_G^*}$  realized by the family of morphisms in  $KK^G$ ,

$$\begin{array}{ccc} \pi_B : \text{Ind}_H^G R_G^H(B) & \xrightarrow{\mu} & c_0(G_H/H) \otimes^{C_0(X)} B \xrightarrow{m \otimes \text{id}} \mathcal{K}(\ell^2(G_H/H)) \otimes^{C_0(X)} B \\ & & \downarrow \equiv \\ & & B \xleftarrow{\cong} \varepsilon(E) \otimes^{C_0(X)} B, \end{array}$$

indexed by  $B$  in  $C_G^*$ . Here,  $\mu$  is the map of Lemma 6.2,  $m : c_0(G_H/H) \rightarrow \mathcal{K}(\ell^2(G_H/H))$  the canonical embedding into the diagonal, which is a  $G$ -equivariant homomorphism, and the vertical arrow is induced by the Morita equivalence of Lemma 7.1. The last isomorphism is by Definition 5.8 and Lemma 5.7.

It is sufficient to show that

$$(9) \quad \pi_{\text{Ind}_H^G(A)} \circ \text{Ind}_H^G(\iota_A) = \text{id}_{\text{Ind}_H^G(A)}$$

in  $KK^G$  and

$$(10) \quad R_G^H(\pi_B) \circ \iota_{R_G^H(B)} = \text{id}_{R_G^H(B)}$$

in  $KK^H$  by [11, IV.1 Theorem 2.(v)].

Now  $\text{Ind}_H^G(\iota_A) : \text{Ind}_H^G(A) \rightarrow \text{Ind}_H^G \mathbf{R}_G^H \text{Ind}_H^G(A)$  is determined by

$$\text{Ind}_H^G(\iota_A)(a) = \sum_{g \in G_H} \sum_{h \in H} 1_g \otimes 1_{hh^*} \otimes 1_h \otimes \sigma_H^{-1}(h^*h) h^*(a(g))$$

for  $a \in \text{Ind}_G^H(A)$ . Then, for  $\mu$  of Lemma 6.2,

$$\begin{aligned} & \mu(\text{Ind}_H^G(\iota_A)(a)) \\ &= \sum_{g \in G_H/H} \sum_{h \in H} 1_{gH} \otimes 1_{ghh^*g^*} \otimes 1_{gh} \otimes \sigma_H^{-1}(h^*h) h^*(a(g)) [g^*g \geq hh^*] \\ &= \sum_{g \in G_H/H} \sum_{h \in H} 1_{gH} \otimes 1_{gh} \otimes a(gh) [g^*g = hh^*] \\ &= \sum_{g \in G_H/H} \sum_{k \in gH} 1_{gH} \otimes 1_k \otimes a(k) \in c_0(G_H/H) \otimes^{C_0(X)} \text{Ind}_H^G(A), \end{aligned}$$

where the second identity uses the isomorphism of Lemma 5.7, and observe that  $\sigma_H^{-1}(h^*h) h^*(a(g)) = a(gh)$  by Definition 4.3.

Now  $\pi_{\text{Ind}_H^G(A)} \circ \text{Ind}_H^G(\iota_A) : \text{Ind}_H^G(A) \rightarrow \text{Ind}_H^G(A)$  is the Kasparov cycle

$$(11) \quad (\rho, \ell^2(G_H/H) \otimes^{C_0(X)} \text{Ind}_H^G(A), 0),$$

where

$$\rho : \text{Ind}_H^G(A) \rightarrow \mathcal{L}(\ell^2(G_H/H) \otimes^{C_0(X)} \text{Ind}_H^G(A))$$

is the multiplication operator

$$\rho(a)(\xi \otimes v) = \sum_{g \in G_H/H} \sum_{k \in gH} \xi(gH) 1_{gH} \otimes 1_k \otimes a(k) v(k)$$

for  $\xi \otimes v \in \ell^2(G_H/H) \otimes^{C_0(X)} \text{Ind}_H^G(A)$ .

Observing the image of  $\rho$ , by a standard argument we may cut down the Hilbert module  $\mathcal{H}$  of the Kasparov cycle (11) to the Hilbert submodule

$$\mathcal{H}_0 = \overline{\text{span}} \left\{ 1_{gH} \otimes \sum_{k \in gH} 1_k \otimes a_k \in \mathcal{H} \mid g \in G_H/H, a_k \in A \right\}$$

and thus obtain an equivalent cycle  $(\rho_0, \mathcal{H}_0, 0)$ . Here  $\rho_0(a) = \rho(a)|_{\mathcal{H}_0}$ . We have an isomorphism

$$u : \mathcal{H}_0 \rightarrow \text{Ind}_H^G(A) : u \left( 1_{gH} \otimes \sum_{k \in gH} 1_k \otimes a_k \right) = \sum_{k \in gH} 1_k \otimes a_k$$

of  $G$ -Hilbert  $\text{Ind}_H^G(A)$ -modules. This transformation yields another equivalent Kasparov cycle  $(i, \text{Ind}_H^G(A), 0)$ , where  $i$  is the multiplication operator on  $\text{Ind}_H^G(A)$ . Hence (9) is verified.

We are going to show (10). We have

$$\iota_{\mathbf{R}_G^H(B)}(b) = \sum_{h \in H} 1_{hh^*} \otimes 1_h \otimes \sigma_H^{-1}(h^*h) h^*(b)$$

for  $b \in \mathbf{R}_G^H(B)$  and

$$\begin{aligned} (12) \quad \mathbf{R}_G^H(\mu)(\iota_{\mathbf{R}_G^H(B)}(b)) &= \sum_{h \in H/H} 1_{hh^*} \otimes 1_{hH} \otimes \sigma_H^{-1}(hh^*)(b) \\ &\in \mathbf{R}_G^H(c_0(G_H/H) \otimes^{C_0(X)} \mathbf{R}_G^H(A)). \end{aligned}$$

Now  $R_G^H(\pi_B) \circ \iota_{R_G^H(B)} : R_G^H(B) \rightarrow R_G^H(B)$  is realized by the Kasparov cycle

$$(13) \quad (\nu, \varepsilon(E_H) \otimes^{C_0(X_H)} \text{Res}_G^H(\ell^2(G_H/H) \otimes^{C_0(X)} R_G^H(B)), 0),$$

where  $\nu(b)$  is the multiplication operator with the element (12). Again, similar as before, we can cut down the Hilbert module of this cycle to

$$\varepsilon(E_H) \otimes^{C_0(X)} \ell^2(H_H/H) \otimes^{C_0(X)} R_G^H(B) \cong R_G^H(B),$$

where the last isomorphism of  $G$ -Hilbert  $R_G^H(B)$ -modules is given by

$$v(hh^* \otimes 1_{hH} \otimes \sigma_H^{-1}(hh^*)(b)) = \sigma_H^{-1}(hh^*)(b).$$

Noticing that  $\sum_{h \in H/H} \sigma_H^{-1}(hh^*)(b) = b$  by Lemma 3.4, we see that the new equivalent Kasparov cycle is  $(j, R_G^H(B), 0)$ , where  $j$  is the multiplication operator. This shows (10).  $\square$

**Example 7.3.** Let us give a simple example where  $G = E$  is finite and consists only of idempotent elements, and  $H = \{e\}$  consists only of a single element of  $G$ . We obtain by direct computation

$$\begin{aligned} KK^E(\text{Ind}_H^E(A), B) &= KK^E((\text{Ind}_H^E(A))_{\varepsilon_e}, B) = KK^E((\text{Ind}_H^E(A))_{\varepsilon_e}, B_{\varepsilon_e}) \\ &= KK^H(A, R_G^H(B)), \end{aligned}$$

verifying Proposition 7.2.

## 8. AN ADAPTION OF A PAPER OF MINGO AND PHILLIPS

In this section we adapt some central results of the paper [15] by Mingo and Phillips to the  $\ell^2(G)$ -space of Definition 5.5. Let  $\mathcal{E}$  be a  $G$ -Hilbert  $B$ -module. Let us write

$$L^2(G, \mathcal{E}) = \ell^2(G) \otimes^{C_0(X)} \mathcal{E}.$$

**Lemma 8.1** (Cf. Lemma 2.3 of [15]). *If  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are  $G$ -Hilbert  $A$ -modules which are isomorphic as Hilbert  $A$ -modules then  $L^2(G, \mathcal{E}_1)$  and  $L^2(G, \mathcal{E}_2)$  are isomorphic as  $G$ -Hilbert  $A$ -modules.*

*Proof.* Let  $u \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$  be a unitary operator. Note that  $gg^* \in \mathcal{L}(\mathcal{E}_i)$  commutes with  $u$  for all  $g \in G$  since  $u$  is  $A$ -linear and  $gg^*(\xi)a = \xi gg^*(a)$  for all  $\xi \in \mathcal{E}_i, a \in A$ . Then it can be checked that  $V : L^2(G, \mathcal{E}_1) \rightarrow L^2(G, \mathcal{E}_2)$  given by  $V(1_g \otimes \xi) := 1_g \otimes gug^*(\xi)$  defines an isomorphism of  $G$ -Hilbert  $A$ -modules. We show that  $V$  is  $G$ -equivariant. For  $h \in G$  we have

$$\begin{aligned} h(V(1_g \otimes \xi)) &= 1_{hg} \otimes h g u g^* h^* h(\xi) [h^* h \geq gg^*] \\ &= V(h(1_g \otimes \xi)), \end{aligned}$$

because  $h^* h \in \mathcal{L}(\mathcal{E}_i)$  commutes with  $gug^* \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ .

For the inner product note that

$$\begin{aligned} \langle V(1_g \otimes \xi), V(1_h \otimes \eta) \rangle &= 1_{gg^*} \otimes \langle gug^*(\xi), h u h^*(\eta) \rangle [h = g] \\ &= \langle 1_g \otimes \xi, 1_h \otimes \eta \rangle \end{aligned}$$

with inner product rules.  $\square$

**Corollary 8.2** (Cf. Theorem 2.4 of [15]). *Let  $\mathcal{E}$  be a  $G$ -Hilbert  $A$ -module which is countably generated and full as a Hilbert  $A$ -module. Then  $(L^2(G, \mathcal{E}))^\infty$  is isomorphic to  $(L^2(G, A))^\infty$  by a  $G$ -equivariant isomorphism of Hilbert  $A$ -modules.*

*Proof.* Same proof as in Mingo and Phillips [15], Theorem 2.4, but by applying Lemma 8.1 instead of [15, Lemma 2.3].  $\square$

**Corollary 8.3** (Cf. Corollary 2.6 of [15]). *Let  $A$  be a  $G$ -algebra and suppose that  $A$  has a strictly positive element. If  $p \in \mathcal{M}(A)$  is a full  $G$ -invariant projection then  $p \otimes 1 \sim 1 \otimes 1$  (Murray–von Neumann equivalence) in  $\mathcal{M}(A \otimes^{C_0(X)} \mathcal{K}(L^2(G)^\infty))$  by a  $G$ -invariant partial isometry.*

*Proof.* The proof of the original goes verbatim through. The usage of the balanced tensor product  $\otimes^{C_0(X)}$  instead of  $\otimes$  is obligatory.  $\square$

**Remark 8.4.** In [3] we considered another model of an  $\ell^2(G)$ -space, denoted  $\widehat{\ell^2}(G)$ , over the  $G$ -algebra  $C_0(X)$  which satisfies analogous results as presented in this section, provided that  $G$  is  $E$ -continuous. This essentially means that certain increasing sequences of projections in  $C_0(X)$  converge pointwise to a projection in  $C_0(X)$ . We could enlarge any inverse semigroup  $G$  to another bigger  $E$ -continuous inverse semigroup  $\overline{G}$  containing  $G$ . (By adjoining to  $G$  the projections corresponding to all subsets of  $X$ .) This would yield another  $\ell^2(G)$ -model, namely  $\text{Res}_{\overline{G}}^G(\widehat{\ell^2}(\overline{G}))$ , now over the  $G$ -algebra  $C_0(X_{\overline{G}})$ . Even if  $\overline{G}$  is usually uncountable, the module  $\widehat{\ell^2}(\overline{G})$  is still countably generated (as  $\varphi_{ge} = \varphi_g \cdot 1_e$  for  $e \in \overline{E}$ ).

## 9. $\varepsilon KK^G$ IS TRIANGULATED

**Definition 9.1.** Let  $\varepsilon KK^G$  denote the full subcategory of  $KK^G$  which is generated by all objects in  $KK^G$  which are isomorphic in  $KK^G$  to a fibered  $G$ -algebra.

In this section we shall show that  $\varepsilon KK^G$  is a triangulated category. At first we need a Cuntz picture of  $\varepsilon KK^G$ . To this end we shall adapt Meyer’s paper [12] which provides a Cuntz picture of Kasparov theory in the group equivariant case.

A central idea of [12] is to make the operator of a Kasparov cycle  $G$ -invariant by switching to the  $L^2(G)$ -version of a Hilbert module, like in 8.1. We want to adapt [12] to our setting, and that is why we need a model of an  $\ell^2(G)$ -space which has nice properties so that the Mingo–Phillips tricks of Section 8 hold. It is less difficult to find such an  $\ell^2(G)$ -space which is a  $G$ -Hilbert module over some  $G$ -algebra  $B$ . By what we observed in Section 8 we could use  $\ell^2(G)$  of Definition 5.5 or  $\text{Res}_{\overline{G}}^G \ell^2(\overline{G})$  of Remark 8.4. But as in [12], one often needs finally to cancel the  $\ell^2(G)$ -space by Morita equivalence as follows. One has given a  $G$ -algebra  $\mathcal{K}(\ell^2(G)) \otimes A$  and wants to get rid of  $\mathcal{K}(\ell^2(G))$ . Hence one uses Morita equivalence  $\mathcal{K}(\ell^2(G)) \equiv \mathcal{K}(B) \cong B$  and so  $KK^G$ -equivalently changes  $\mathcal{K}(\ell^2(G)) \otimes A$  to  $B \otimes A$ . Now  $B$  must be the neutral element for the tensor product, like  $B = \mathbb{C}$ , such that  $B \otimes A \cong A$ . That is why we have this constraint on the coefficient algebra  $B$  of the  $\ell^2(G)$ -module.

The  $\ell^2(G)$ -space of Definition 5.5 over the  $G$ -algebra  $B = \varepsilon(E)$  satisfies this constraint for the class of fibered  $G$ -algebras  $A$  and the balanced tensor product by Lemma 5.7.

**Lemma 9.2.** (a) *If  $T, A, B$  are  $G$ -algebras and  $T$  is equipped with the trivial  $G$ -action then there is a canonical isomorphism*

$$(T \otimes A) \otimes^{C_0(X)} B \cong T \otimes (A \otimes^{C_0(X)} B).$$

(b) *If  $A, B, C$  are  $G$ -algebras and  $f : A \rightarrow B$  is a  $G$ -equivariant homomorphism then*

$$\text{cone}(f) \otimes^{C_0(X)} C \cong \text{cone}(f \otimes^{C_0(X)} \text{id}_C).$$

**Corollary 9.3.** *The class of fibered  $G$ -algebras is closed under taking suspension and cones.*

*Proof.* Set  $T = \Sigma = C_0(\mathbb{R})$  (suspension) and  $B = C = \varepsilon(E)$  in the last lemma.  $\square$

Our aim is to slightly adapt [3, Section 5] by making the following simple modifications:

- Replace every occurrence of the  $G$ -Hilbert  $C_0(X)$ -module  $\widehat{\ell}^2(G)$  in [3, Section 5] by the  $G$ -Hilbert  $\varepsilon(E)$ -module  $\ell^2(G)$  of Definition 5.5.
- Substitute every single occurring  $C_0(X)$  in [3, Section 5] which appears as a  $G$ -algebra in its own right (not in  $\otimes^{C_0(X)}$ ) by the  $G$ -algebra  $\varepsilon(E)$  (for example in  $C_0(X)^\infty$  or in  $C_0(X) \oplus \mathcal{H}$ ).

By an analogous replacement  $L^2(G) \rightarrow \widehat{\ell}^2(G)$ ,  $\mathbb{C} \rightarrow C_0(X)$  and  $\otimes \rightarrow \otimes^{C_0(X)}$ , Section 5 of [3] was obtained from Meyer’s paper [12].

In the next theorem we state a version of [12, Theorem 6.5] in a slightly simplified but less technical form, which summarizes its quintessence.

Since we did not go through all details of the paper [12], we should view the following theorem as a conjecture!

**Theorem 9.4** (Cf. [12]). *Let  $A$  and  $B$  be fibered  $G$ -algebras and  $x \in KK^G(A, B)$ . Then there exist fibered  $G$ -algebras  $A'$  and  $B'$ , isomorphisms  $a \in KK^G(A, A')$  and  $b \in KK^G(B, B')$ , and a  $G$ -equivariant  $*$ -homomorphism  $f : A' \rightarrow B'$  such that*

$$x = a \circ f \circ b^{-1}.$$

**Proposition 9.5** (Cf. [14]). *The category  $\varepsilon KK^G$  is triangulated by calling a triangle distinguished if it is isomorphic to a mapping cone triangle, and by defining the translation functor to be  $A \mapsto \Sigma^{-1}A$  (desuspension).*

For the details of the last proposition see [3, Section 6] and [14], respectively. Notice also, that actually a certain direct limit  $\widetilde{\varepsilon KK^G}$  induced by the suspension functor is triangulated, and  $\varepsilon KK^G$  is just its sloppy notation. Both categories, however, are equivalent.

## 10. THE BAUM–CONNES MAP

Throughout this section the fibered restriction functors and induction functors are understood to be restricted to the category  $\varepsilon KK^G$ . That is we view induction and fibered restriction as  $\text{Ind}_H^G : KK^H \rightarrow \varepsilon KK^G$  and  $R_G^H : \varepsilon KK^G \rightarrow KK^H$ .

To avoid cumbersome notation, we shall make the following convention:

*Throughout this section we shall exclusively work with the triangulated category  $\varepsilon KK^G$  but denote it by  $KK^G$  for simplicity most of the time!*

**Definition 10.1.** An exact functor  $F : \mathcal{S} \rightarrow \mathcal{T}$  between triangulated categories  $\mathcal{S}$  and  $\mathcal{T}$  is a suspension-intertwining ( $F \circ S_{\mathcal{S}} = S_{\mathcal{T}} \circ F$ ) functor which sends exact sequences  $SB \rightarrow C \rightarrow A \rightarrow B$  canonically to exact sequences  $SF(B) \rightarrow F(C) \rightarrow F(A) \rightarrow F(B)$  (see [10]). A functor  $F$  between triangulated categories is called *triangulated* if it is exact and additive.

**Lemma 10.2.** *The fibered restriction functors  $R_G^H$  and induction functors  $\text{Ind}_H^G$  are triangulated functors. Also for every  $G$ -algebra  $B$ , the balanced tensor product functor  $A \mapsto A \otimes^{C_0(X)} B$  from the category  $KK^G$  into itself is triangulated.*

*Proof.* Given an exact triangle in  $KK^H$ , we may switch to its isomorphic mapping cone triangle according to definition [3, 6.4]. This mapping cone triangle is sent canonically to a mapping cone triangle (and hence exact triangle) by the fibered restriction and induction functors by Lemma 9.2 and [3, 4.3].  $\square$

For the orthogonal subcategory  $\mathcal{S}^\perp$  of a triangulated subcategory  $\mathcal{S}$  see [10, 4.8]. The expression  $\langle \mathcal{S} \rangle$  denotes the generated triangulated subcategory of a subcategory  $\mathcal{S}$  of a triangulated category.

**Definition 10.3** (Cf. Definition 4.1 of [14]). An object  $A$  in  $KK^G$  is called *compactly induced* if there exists an object  $B$  in  $KK^G$  and a finite subinverse semigroup  $H \subseteq G$  such that  $A$  is isomorphic to  $\text{Ind}_H^G(B)$  in  $KK^G$ . The full subcategory of  $KK^G$  induced by the compactly induced objects is denoted by  $\mathcal{CI}$ .

**Definition 10.4** (Cf. Definition 4.1 of [14]). A morphism  $f \in KK^G(A, B)$  is called a *weak equivalence* if  $R_G^H(f)$  is invertible in  $KK^H$  for all finite subinverse semigroups  $H \subseteq G$ .

**Definition 10.5** (Cf. Definition 4.5 of [14]). A  $\mathcal{CI}$ -simplicial approximation of an object  $A$  in  $KK^G$  is a weak equivalence  $f \in KK^G(B, A)$  such that  $B$  is an object in  $\langle \mathcal{CI} \rangle$ .

**Definition 10.6** (Cf. Definition 4.5 of [14]). A *Dirac morphism* is a  $\mathcal{CI}$ -simplicial approximation of  $\varepsilon(E)$ .

**Definition 10.7** (Cf. Definition 4.1 in [14]). Call an object  $A$  in  $KK^G$  *weakly contractible* if  $R_G^H(A) = 0$  in  $KK^H$  for all finite subinverse semigroups  $H \subseteq G$ . Write  $\mathcal{CC}$  for the full subcategory of  $KK^G$  of weakly contractible objects.

**Lemma 10.8** (Cf. Proposition 4.4 of [14]). *We have  $\mathcal{CC} = \langle \mathcal{CI} \rangle^\perp$ .*

*Proof.* One proves this verbatim as in [14, Proposition 4.4]. One just needs the adjointness relation of Proposition 7.2.  $\square$

**Lemma 10.9** (Cf. Lemma 4.2 of [14]). *The categories  $\langle \mathcal{CI} \rangle$  and  $\mathcal{CC}$  are localizing subcategories of  $KK^G$ . They are closed under forming the balanced tensor product  $A \mapsto A \otimes^{C_0(X)} B$  for all  $G$ -algebras  $B$ .*

*Proof.* One proves this like Lemma 4.2 of [14]. The stability under tensor products follows from Lemmas 6.3 and 6.4.  $\square$

**Theorem 10.10** ([13]). *There exists a Dirac morphism. Even more, every object in  $\varepsilon KK^G$  has a  $\mathcal{CI}$ -simplicial approximation.*

*Proof.* One proceeds verbatim as in the first two paragraphs of the proof of [13, Theorem 7.3], which handles the discrete group case. One just replaces the ordinary restriction functor  $\text{Res}_H^G$  with the fibered restriction functor  $R_H^G$  everywhere. It is required that this functor commutes with direct sums, which is satisfied. Also the necessary fact that the functor  $R_G^H$  is right adjoint to the functor  $\text{Ind}_H^G$  is verified in Proposition 7.2. The claim follows then by the verbatim analogous argument given in the paragraph after [13, Theorem 7.3].  $\square$

Even if Theorem 10.10 shows already that every object allows a  $\mathcal{CI}$ -simplicial approximation, we shall also demonstrate how this fact can already be deduced from the existence of a Dirac morphism by tensoring with a coefficient algebra. This is the next lemma and its corollary.

**Lemma 10.11** (Cf. Theorem 4.7 of [14]). *Let  $D \in KK^G(P, \varepsilon(E))$  be a Dirac morphism with  $P \in \langle \mathcal{CI} \rangle$ . Then there exists an exact triangle*

$$(14) \quad P \xrightarrow{D} \varepsilon(E) \longrightarrow N \longrightarrow \Sigma^{-1}P$$

*in  $KK^G$  with  $N \in \mathcal{CC}$ . For every fibered  $G$ -algebra  $A$  this induces canonically by tensoring an exact triangle*

$$(15) \quad P \otimes_{C_0(X)} A \xrightarrow{D \otimes id} \varepsilon(E) \otimes_{C_0(X)} A \longrightarrow N \otimes_{C_0(X)} A \longrightarrow \Sigma^{-1}(P \otimes_{C_0(X)} A)$$

*in  $KK^G$ . Here, one has  $P \otimes_{C_0(X)} A \in \langle \mathcal{CI} \rangle$  and  $N \otimes_{C_0(X)} A \in \mathcal{CC}$ .*

*Proof.* By the axioms of a triangulated category, the morphism  $D$  from  $KK^G$  fits into an exact triangle as in (14) for some object  $N$  in  $KK^G$ . Since  $R_G^H$  is an exact functor by Lemma 10.2, this triangle canonically induces exact triangles in  $KK^H$  via  $R_G^H$  for all finite subinverse semigroups  $H$  in  $G$ . By Definition 10.6,  $R_G^H(D)$  is an isomorphism in  $KK^H$ , and so  $R_G^H(N)$  vanishes in  $KK^H$  by Corollary 1.2.4 and Remark 1.1.21 of [16], or confer [14, Lemma 2.2]. But this means that  $N \in \mathcal{CC}$ .

By Lemma 10.2 the sequence (15) is exact. The last claim follows from Lemma 10.9.  $\square$

**Remark 10.12.** The importance of Lemma 10.11 is that its validity is equivalent to the existence of an exact localization functor  $L : KK^G \rightarrow KK^G$  with kernel  $\langle \mathcal{CI} \rangle$ , see for example Proposition 4.9.1 in [10]. This implies the existence of an exact colocalization functor  $\Gamma : KK^G \rightarrow KK^G$  with kernel  $\mathcal{CC}$  and an equivalence  $KK^G/\mathcal{CC} \cong \langle \mathcal{CI} \rangle$  in the opposite category of  $KK^G$ , see for example Propositions 4.12.1, 4.10.1 and 4.11.1 in [10] together with Lemma 10.8. Confer also Proposition 2.9 and the remarks after Definition 4.2 in [14]. The complementarity condition of [14, Definition 2.8] is satisfied by [10, Proposition 4.10.1] and Lemma 10.8, which combine to  $\text{Im} L = \langle \mathcal{CI} \rangle^\perp = \mathcal{CC}$ .

**Corollary 10.13.** *Every object  $A$  in  $\varepsilon KK^G$  has a  $\mathcal{CI}$ -simplicial approximation, for example  $D \otimes id$  of (15).*

*Proof.* Let  $D \in KK^G(P, \varepsilon(E))$  be a Dirac morphism as in Lemma 10.11. Since by Definition 10.6  $R_G^H(D)$  is an isomorphism in  $KK^H$  for every finite subinverse semigroup  $H \subseteq G$ ,  $R_G^H(D \otimes id) = R_G^H(D) \otimes R_G^H(id)$  (see Lemma 6.3) is also an isomorphism. Hence  $D \otimes id$  is a weak equivalence.  $\square$

**Definition 10.14** (Baum–Connes map). Let  $A$  be an object in  $\varepsilon KK^G$ . (That is,  $A$  is a  $G$ -algebra which is isomorphic in  $KK^G$  to a fibered  $G$ -algebra.) Choose a  $\mathcal{CI}$ -simplicial approximation  $D \in KK^G(\tilde{A}, A)$  for  $A$ . (That is,  $\tilde{A}$  is a kind of an approximation of  $A$  which in the easiest case is an induced algebra  $\tilde{A} = \text{Ind}_H^G(B)$ .) For a given descent functor  $j^G : KK^G \rightarrow KK$  (there are several choices corresponding to different crossed products) form the morphism

$$j^G(D) \in KK(\tilde{A} \rtimes G, A \rtimes G)$$

as a potentially good approximation between  $\tilde{A} \rtimes G$  and  $A \rtimes G$ .

The *Baum–Connes map* with coefficient algebra  $A$  (and with respect to the descent functor  $j^G$ ) is defined to be the abelian group homomorphism  $\nu^G : K(\tilde{A} \rtimes$



$G) \rightarrow K(A \rtimes G)$  as indicated in the following commuting diagram:

$$\begin{array}{ccc}
 K(\tilde{A} \rtimes G) & \xrightarrow{\nu^G} & K(A \rtimes G) \\
 \downarrow \cong & & \uparrow \cong \\
 KK(\mathbb{C}, \tilde{A} \rtimes G) & \xrightarrow{\otimes_{j^G(D)}} & KK(\mathbb{C}, A \rtimes G)
 \end{array}$$

The vertical arrows are the usual isomorphisms [7, §6.3] and the bottom arrow is the map which takes the Kasparov product with  $j^G(D)$ .

Definition 10.14 does not depend on the choice of the  $\mathcal{CI}$ -simplicial approximation  $D$ , see Proposition 2.9.2 of [14].

**Remark 10.15.** (a) Notice that in the case of Sieben's crossed product, the domain  $K(\tilde{A} \rtimes G)$  of the Baum–Connes map is potentially computable by homological means in a triangulated category as explained in the introduction by (1), Theorem 4.5 and [13, Theorem 5.1] applied to the functor  $F(A) = K(A \rtimes G)$  from  $KK^G$  to the abelian groups.

(b) For the full crossed product the constructed Baum–Connes map is usually less powerful. Indeed, for instance if  $E$  has a minimal element  $e_0$  then  $\mathbb{C}$  is a fibered  $G$ -algebra. (If  $E$  has no minimal element it may be adjoined to  $G$  by setting  $G' := G \sqcup \{e_0\}$  and  $e_0g = ge_0 = e_0$ .) Note that  $e_0G = e_0Ge_0$  is a subgroup of  $G$  and  $KK^G(\mathbb{C}, A) = KK^{e_0G}(\mathbb{C}, A_{\varepsilon_{e_0}})$ . Hence, if  $e_0G$  is the trivial group then  $\text{Ind}_{\{e_0\}}^G(\mathbb{C}) = \mathbb{C}$  is a Dirac element and the Baum–Connes map  $K(\tilde{\mathbb{C}} \rtimes G) \rightarrow K(\mathbb{C} \rtimes G)$  is trivially the identity map. In any case, at the end of the day we miss a Green imprimitivity theorem as in Theorem 4.5 for potential computation of the domain of the Baum–Connes map.

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